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ASYMPTOTIC ANALYSIS OF THREE-DIMENSIONAL DYNAMICAL ELASTIC EQUATIONS FOR A THIN MULTILAYER ANISOTROPIC PLATE OF ARBITRARY STRUCTURE†

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The asymptotic method of [1–4] is used to derive two-dimensional dynamical equations for a plate made up of N anisotropic layers with planes of elastic symmetry parallel to the faces. It is shown that, unlike the case of isotropically layered plates [4], these equations do not reduce to the equations for an equivalent monoplate and Kirchhoff's first hypothesis is inapplicable. The full stress tensor, including asymptotically small components, is determined, as are the integral characteristics of the assembly. The general cases of separation of the problems of bending and longitudinal tension-compression-shearing are analysed, and some particular anisotropic structures are considered. It is proposed to solve static problems by representing the unknown functions as functions of a complex variable.

1. CONSIDER a layered plate consisting of N anisotropic linearly elastic layers, perfectly bonded together along horizontal planes of contact and such that there is a horizontal plane of elastic symmetry at each point. This will be the case, for example, in a plate formed by stacking unidirectional composites with rotated horizontal principal axes. The positions of the layers are represented by the Cartesian coordinates

$$X_1,\ X_2 \in \Omega \subset R^2,\quad Z_j \leqslant X_3 \equiv Z \leqslant Z_{j+1}\quad (j=1,2,\ldots,N)$$

Let $H_j = Z_{j+1} - Z_j$ be the thickness of the jth layer, ρ_j its density, G_j its stiffness matrix, H_0, ρ_0, E_0, $c_0 = (E_0/\rho_0)^{1/2}$, T_0 are the half-thickness and characteristic density, modulus of elasticity, velocity and duration of dynamical processes, respectively, for the entire assembly. Assuming that the variability of the stress–strain state in the longitudinal direction is determined by a minimal characteristic dimension L_0 (depending on the external forces and the geometry of the plate), we shall always use dimensionless variables, displacements, stresses and stiffness matrix

$$(x_1,x_2)=\frac{(X_1,\ X_2)}{L_0},\quad z=\frac{Z}{H_0},\quad t=\frac{T}{T_0},\quad \mathbf{g}=\frac{\mathbf{G}}{E_0}$$

$$(\mathbf{u},w)\equiv(u_1,u_2,w)=\frac{\mathbf{U}}{H_0},\quad \sigma_{\alpha\beta}=\frac{\Sigma_{\alpha\beta}}{E_0}$$

A superscript j, wherever necessary, will denote the number of the layer. The external loads are concentrated on the exposed faces of the plate

$$\sigma_3=\sigma^{\mp}(x_1,x_2,t),\quad \tau\equiv(\sigma_{13},\sigma_{23})=\tau^{\mp}(x_1,x_2,t)\qquad(1.1)$$
$$(z^{\mp}=z_1,\ z_{N+1})$$

and are functions varying at a fairly slow rate. The displacements and stresses are continuous across the interfaces

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$$\sigma_{\alpha 3}^{j} = \sigma_{\alpha 3}^{j-1}, \quad (\mathbf{u}, w)^{j} = (\mathbf{u}, w)^{j-1}, \quad z = z_j, \quad j = 2, 3, \ldots, N \tag{1.2}$$

We shall treat the quotient $\epsilon = H_0/L_0$ as a small parameter, $T_0 = \epsilon^{\tau-1}L_0/c_0$ (later we shall consider the case $\tau = 0$); the ratios of the layer thicknesses, stiffnesses and densities do not form new small (or large) parameters.

The three-dimensional dynamical elastic equations for each layer may be written as follows:

$$\begin{aligned}
&\partial_z^2 w + \epsilon\partial_z \mathbf{M}_3\mathbf{u} + \epsilon^2 N_3 w - \rho(\rho_0 g_{33})^{-1}\partial_t^2 w = 0 \\
&\partial_z^2 \mathbf{K}_1\mathbf{u} + \epsilon\partial_z M_1 w + \epsilon^2 \mathbf{N}_1\mathbf{u} - \rho(\rho_0 g_{55})^{-1}\partial_t^2 u_1 = 0 \quad (1 \leftrightarrow 2) \\
&g_{33}\mathbf{M}_3 = \mathbf{i}_1\{(g_{13}+g_{55})\partial_1 + (g_{45}+g_{36})\partial_2\} + \mathbf{i}_2\{(g_{45}+g_{36})\partial_1 + (g_{23}+g_{44})\partial_2\} \\
&g_{33}N_3 = g_{55}\partial_1^2 + 2g_{45}\partial_{12}^2 + g_{44}\partial_2^2 \\
&g_{55}\mathbf{K}_1 = g_{55}\mathbf{i}_1 + g_{45}\mathbf{i}_2, \quad g_{55}M_1 = (g_{13}+g_{55})\partial_1 + (g_{45}+g_{36})\partial_2 \\
&g_{55}\mathbf{N}_1 = \mathbf{i}_1\{g_{11}\partial_1^2 + 2g_{16}\partial_{12}^2 + g_{66}\partial_2^2\} + \mathbf{i}_2\{g_{16}\partial_1^2 + (g_{12}+g_{66})\partial_{12}^2 + g_{26}\partial_2^2\}
\end{aligned} \tag{1.3}$$

To solve this system of equations, we expand the unknown functions in asymptotic series in powers of ϵ^s [1, 2, 5] in each layer ($s = 0, 1, 2, \ldots$)

$$\mathbf{u} = \epsilon^{\lambda+1}\Sigma\epsilon^s\mathbf{u}^{(s)}, \quad w = \epsilon^{\lambda}\Sigma\epsilon^s w^{(s)} \tag{1.4}$$

thereby obtaining the equations for the components of the displacements

$$\begin{aligned}
&\partial_z^2 w^{(s)} + \partial_z\mathbf{M}_3\mathbf{u}^{(s-2)} + N_3 w^{(s-2)} - \rho(\rho_0 g_{33})^{-1}\partial_t^2 w^{(s-4+2\tau)} = 0 \\
&\partial_z^2\mathbf{K}_1\mathbf{u}^{(s)} + \partial_z M_1 w^{(s)} + \mathbf{N}_1\mathbf{u}^{(s-2)} - \rho(\rho_0 g_{55})^{-1}\partial_t^2 u_1^{(s-4+2\tau)} = 0
\end{aligned} \tag{1.5}$$

2. Integrating system (1.5) for $s = 0$, 1 and assuming that $\tau < 2$, we obtain expansions of the s-components in terms of the variable z

$$\begin{aligned}
&w^{(s)} = w_0^{(s)} + zw_1^{(s)}, \quad \mathbf{u}^{(s)} = \mathbf{u}_0^{(s)} + z\mathbf{u}_1^{(s)} + z^2/2\,\mathbf{u}_2^{(s)} \\
&2\mathbf{K}_\beta\mathbf{u}_2^{(s)} = -M_\beta w_1^{(s)}, \quad \sigma_\alpha^{(s)} = g_{\alpha 3}w_1^{(s)} \quad (\alpha = 1, 2, 3;\ \beta = 1, 2) \\
&\sigma_{13}^{(s)} = g_{45}[u_{21} + \partial_2 w_0]^{(s)} + g_{55}[u_{11} + \partial_1 w_0]^{(s)} + 2z[g_{45}u_{22} + g_{55}u_{12}]^{(s)} \quad (1 \leftrightarrow 2) \\
&\sigma_{12}^{(s)} = \begin{cases} g_{36}w_1^{(s)}, & g_{36} \neq 0 \\ (g_{16}\partial_1 + g_{66}\partial_2)u_1^{(s)} + (g_{66}\partial_1 + g_{26}\partial_2)u_2^{(s)}, & g_{36} = 0 \end{cases}
\end{aligned} \tag{2.1}$$

In view of the necessary independence of the stresses σ_α and conditions (1.2), we conclude that, irrespective of the layer index j,

$$\mathbf{u}^{(1)} = \mathbf{u}_0^{(s)} - z\,\mathrm{grad}\,w_0^{(s)}, \quad \sigma_\alpha^{(s)} = \sigma_{\beta 3}^{(s)} = w_1^{(s)} = u_{\beta 2}^{(s)} \equiv 0$$

and also

$$\sigma_{12}^{(j,s)} = \begin{cases} 0, & g_{36}^j \neq 0 \\ [\mathbf{i}_1(g_{16}\partial_1 + g_{66}\partial_2) + \mathbf{i}_2(g_{66}\partial_1 + g_{26}\partial_2)]_j(\mathbf{u}_0 - z\,\mathrm{grad}\,w_0)^{(s)}, & g_{36}^j = 0 \end{cases}$$

The functions on the right of these equalities depend only on x_1, x_2, t. In general, for expansions of the s-components in powers of z, that is,

$$\mathbf{u}^{(s)} = \sum_{k=0}^{K} z^k\mathbf{u}_k^{(s)}, \quad w^{(s)} = \sum_{k=0}^{K} z^k w_k^{(s)}, \quad K = 2\left[\frac{s}{2}\right]$$

we have the following recurrence relations in each layer

$$\begin{aligned}
&(k+2)(k+1)w_{k+2}^{(s)} + (k+1)\mathbf{M}_3\mathbf{u}_{k+1}^{(s-2)} + N_3 w_k^{(s-2)} - \rho(\rho_0 g_{33})^{-1}\partial_t^2 w_k^{(s-4+2\tau)} = 0 \quad (2.2) \\
&(k+2)(k+1)\mathbf{K}_1\mathbf{u}_{k+2}^{(s)} + (k+1)M_1 w_{k+1}^{(s)} + \mathbf{N}_1\mathbf{u}_k^{(s-2)} - \rho(\rho_0 g_{55})^{-1}\partial_t^2 u_{1k}^{(s-4+2\tau)} = 0 \quad (1 \leftrightarrow 2)
\end{aligned}$$

3. In our analysis of the components with indices $s+2=2, 3$ and their subsequent values, we take $\tau=0$ ($T \sim L_0/c_0\epsilon$ and the frequency in the harmonic problem is $\omega_0 \sim 2\pi c_0\epsilon/L_0$). We are thus not considering the version of the quasistatic equations for all the displacements, or the appearance in all equations of inertial terms ($\tau>0$, when short waves are excited and the condition $\epsilon \ll 1$ is violated). We will confine ourselves to the classical Kirchhoff–Love long-wave theory of plates. The recurrence relations (2.2) imply that in each layer

$$
\begin{aligned}
&2w_2^{(s+2)} = L_2 w_0^{(s)}, \quad 6\mathbf{K}_\beta \mathbf{u}_3^{(s+2)} = (\mathbf{N}_\beta \operatorname{grad} - M_\beta L_2)\, w_0^{(s)} \\
&2\mathbf{K}_\beta \mathbf{u}_2^{(s+2)} = -M_\beta w_1^{(s+2)} - \mathbf{N}_\beta \mathbf{u}_0^{(s)}, \quad \sigma_{31}^{(s+2)} = 0 \\
&L_2 = \mathbf{M}_3 \operatorname{grad} - N_3 = \mathbf{L}_1 \operatorname{grad} \\
&g_{33}\mathbf{L}_1 = \mathbf{i}_1(g_{13}\partial_1 + g_{36}\partial_2) + \mathbf{i}_2(g_{36}\partial_1 + g_{23}\partial_2)
\end{aligned}
\tag{3.1}
$$

Since the boundary conditions (1.1) are arbitrary, it remains to require that

$$
\sigma_{30}^{(s+2)} = 0, \quad w_1^{(s+2)} = -\mathbf{L}_1 \mathbf{u}_0^{(s)}, \quad 2\mathbf{K}_\beta \mathbf{u}_2^{(s+2)} = [\mathbf{M}_\beta \mathbf{L}_1 - \mathbf{N}_\beta]\, \mathbf{u}_0^{(s)}
\tag{3.2}
$$

Let us determine the structure of the other stresses. By Hooke's law and (3.1) and (3.2), we obtain

$$
\begin{aligned}
&\sigma_1^{(j,s+2)} = \sigma_{10}^{(j,s+2)} + z\sigma_{11}^{(j,s+2)} = \mathbf{d}_1(\gamma_{pq}^j)\,\mathbf{u}_0^{(s)} - \mathbf{d}_1(z\gamma_{pq}^j) \operatorname{grad} w_0^{(s)} \\
&\sigma_{12}^{(j,s+2)} = \sigma_{120}^{(j,s+2)} + z\sigma_{121}^{(j,s+2)} = \mathbf{d}(\gamma_{pq}^j)\,\mathbf{u}_0^{(s)} - \mathbf{d}(z\gamma_{pq}^j) \operatorname{grad} w_0^{(s)} \\
&\mathbf{d}_1(\gamma_{pq}) \equiv \mathbf{i}_1(\gamma_{11}\partial_1 + \gamma_{16}\partial_2) + \mathbf{i}_2(\gamma_{16}\partial_1 + \gamma_{12}\partial_2) \quad (1 \leftrightarrow 2) \\
&\mathbf{d}(\gamma_{pq}) \equiv \mathbf{i}_1(\gamma_{16}\partial_1 + \gamma_{66}\partial_2) + \mathbf{i}_2(\gamma_{66}\partial_1 + \gamma_{26}\partial_2), \quad \gamma_{pq} \equiv g_{pq} - g_{p3}g_{3q}/g_{33}
\end{aligned}
\tag{3.3}
$$

defining $\sigma_{\beta 3}^{(j,s+2)}$ using a different procedure. Indeed, since the surface load is independent of thickness and the contact stresses (1.1) and (1.2) are equal across the layer interfaces, it follows that

$$
\begin{aligned}
&\sigma_{\beta 30}^{(j,s+2)} = \frac{1}{2}\Big\{(\tau_\beta^+ + \tau_\beta^-)\,\delta_{\lambda+s+3}^0 + \Big[\sum_{n=1}^{j-1} - \sum_{j+1}^{N}\Big] \sum_{k=1}^{K} (z_{n+1}^k - z_n^k)\,\sigma_{\beta 3k}^{(n,s+2)} - \\
&- \sum_{k=1}^{K} (z_{j+1}^k + z_j^k)\,\sigma_{\beta 3k}^{(j,s+2)}\Big\} \\
&\sum_{j=1}^{N} \sum_{k=1}^{K} (z_{j+1}^k - z_j^k)\,\sigma_{\beta 3k}^{(j,s+2)} = (\tau_\beta^+ - \tau_\beta^-)\,\delta_{\lambda+s+3}^0
\end{aligned}
\tag{3.4}
$$

The leading components $\sigma_{\beta 3k}^{(s+2)}$ are

$$
\begin{aligned}
&\sigma_{131}^{(s+2)} = \mathbf{a}_1 \mathbf{u}_0^{(s)}, \quad \sigma_{132}^{(s+2)} = \tfrac{1}{2} b_1 w_0^{(s)} \quad (1 \leftrightarrow 2) \\
&\mathbf{a}_1(\gamma_{pq}) = g_{55}(M_1\mathbf{L}_1 - \mathbf{N}_1) - l_1\mathbf{L}_1, \quad b_1 = -\mathbf{a}_1 \operatorname{grad}, \quad l_1 = g_{55}\partial_1 + g_{45}\partial_2
\end{aligned}
$$

and after these are substituted into (3.4), we obtain the final expressions for the stresses and a compatible quasi-static system of equations for the longitudinal and transverse stresses.

Since $\sigma_3^{(s+2)} = 0$ for $s = 0, 1$ and the surface load is independent of ϵ, it is natural to set $\lambda = -4$ and to check the expansions corresponding to indices $s+4 = 4, 5$. In the relations analogous to (3.4) for the normal stresses, we must make the following substitutions: τ_β for σ, $\sigma_{\beta 3}$ for σ_3 and K for $K-1$, and take the Kronecker delta $\delta_{\lambda+s+4}^0$. Following Hooke's law and taking the recurrence relations (2.2) into account, we determine (for all $s \geqslant 0$, $k>0$) the leading components of the normal stresses in each layer

$$
\sigma_{3k}^{(s+4)} = \frac{1}{k}\left\{\frac{\rho}{\rho_0}\,\partial_t^2 w^{(s+2\tau)} - \operatorname{div}\tau^{(s+2)}\right\}_{k-1}
$$

Substituting these equalities, we obtain an expression for $\sigma_3^{(j,s+4)}$ and a third mixed equation for the normal and longitudinal displacements. Omitting the cumbersome algebra and the simplifications of double sums, we present the final result

$$\sigma_{\beta 3}^{(j,\,s+2)} = \tau_\beta^\pm \delta_s^1 \mp \Sigma_\pm \Big[h_n \mathbf{a}_\beta^n \mathbf{u}_0 + \frac{z_{n+1}^2 - z_n^2}{2} b_\beta^n w_0\Big]^{(s)} + \tag{3.5}$$
$$+ (z - z_j^\pm)\, \mathbf{a}_\beta^j \mathbf{u}_0^{(s)} + \frac{z^2 - (z_j^\pm)^2}{2} b_\beta^j w_0^{(s)}$$

$$\sigma_3^{(j,\,s+4)} = \sigma^\pm \delta_s^0 + (z^\pm - z)\,\mathrm{div}\,\tau^\pm - \Big[(z_j^\pm - z)\frac{\rho_j}{\rho_0} \pm \Sigma_\pm \frac{\rho_n h_n}{\rho_0}\Big] \partial_t^2 w_0^{(s)} \pm$$
$$\pm \Sigma_\pm \left\{ \Big[h_n - \frac{z_{n+1}^2 - z_n^2}{2}\Big] \mathbf{a}_*^n \mathbf{u}_0 + \Big[\frac{z_{n+1}^2 - z_n^2}{2} - \frac{z_{n+1}^3 - z_n^3}{3}\Big] b_*^n w_0 \right\}^{(s)} -$$
$$- \frac{(z - z_j^\pm)^2}{2} \mathbf{a}_*^j \mathbf{u}_0^{(s)} - \Big[\frac{z^3}{6} - \frac{z (z_j^\pm)^2}{2} + \frac{(z_j^\pm)^3}{3}\Big] b_*^j w_0^{(s)}$$

$$\left\{ \Sigma_- = \sum_{n=1}^{j-1}, \quad \Sigma_+ = \sum_{n=j+1}^{N}, \quad z_j^\pm = z_{j+1},\ z_j \right\}$$

$$\rho_* \partial_t^2 w_0^{(s)} + \mathbf{A}_* \mathbf{u}_0^{(s)} + B_* w_0^{(s)} = (\sigma^+ - \sigma^-)\,\delta_s^0 + \mathrm{div}(z^+ \tau^+ - z^- \tau^-)\,\delta_s^1 \tag{3.6}$$
$$\mathbf{A}_\beta \mathbf{u}_0^{(s)} + B_\beta w_0^{(s)} = (\tau_\beta^+ - \tau_\beta^-)\,\delta_s^1$$
$$\mathbf{a}_* \equiv \partial_1 \mathbf{a}_1 + \partial_2 \mathbf{a}_2 = -\mathbf{i}_1 b_1 - \mathbf{i}_2 b_2, \quad b_* \equiv \partial_1 b_1 + \partial_2 b_2 = -\mathbf{a}_* \mathrm{grad}$$

$$\mathbf{A}_\beta \equiv \mathbf{a}_\beta(D_{pq}^1), \quad B_\beta \equiv b_\beta(D_{pq}^2), \quad \mathbf{A}_* \equiv \mathbf{a}_*(D_{pq}^2), \quad B_* \equiv b_*(D_{pq}^3) \tag{3.7}$$
$$D_{pq}^n \equiv \frac{1}{n} \sum_{j=1}^{N} (z_{j+1}^n - z_j^n)\,\gamma_{pq}^j \quad (n = 1, 2, 3;\ pq = 11, 12, 22, 16, 66, 26)$$
$$\rho_* = \sum_{j=1}^{N} \frac{h_j \rho_j}{\rho_0}$$

Expressions (3.5) and (3.3) are the first two terms of the asymptotic expansions of all the components of the stress tensor; a compatible system of equations for the displacements is given by (3.6) and (3.7); it is accurate to within terms $O(\epsilon^2)$.

4. We will now indicate the orders of the main (dimensional) physical quantities that correspond to the above choice of asymptotic expansions. Their behaviour as $\epsilon \to +0$ is similar to the case of a homogeneous plate [1]. They all have the form

$$V = M\epsilon^\mu \{ v^{(0)} + \epsilon v^{(1)} + O(\epsilon^2) \}$$

where $r = 0$ corresponds to the problem of the bending of the plate by a normal load and $r = 1$ corresponds to the reaction of the plate to a tangential load. The values $M = L_0$ and $\mu = -3$, $\mu = -2$ are obtained for the transverse and longitudinal displacements, $M = E_0$ and $\mu = -2$ for stresses $\alpha\beta = 11, 12, 22$ and $\mu = -1$, $\mu = 0$ for $\alpha\beta = 13, 23$ and $\alpha\beta = 33$ (second-degree stresses are not determined in hypothetical plate theories).

Considering the bending moments $m_{\alpha\beta}$ ($M = E_0 L_0^2$) and the linear and transverse forces $q_{\alpha\beta}$, $q_{\beta 3}$ ($M = E_0 L_0$) obtained by integrating the stresses over the plate cross section as a whole, we see that the dimensional quantities are smaller than the stresses by two and one orders of magnitude, respectively. The expressions for the bending moments and linear forces are analogous to (3.3), except that the arguments γ_{pq}, $z\gamma_{pq}$ of the operators are replaced by the membrane-flexural and flexural stiffnesses D_{pq}^2, D_{pq}^3 for the moments or by the membrane and membrane-flexural stiffnesses D_{pq}^1, D_{pq}^2 for the forces. Expressions for the transverse forces are determined by the equality

$$q_{\beta 3}^{(s+2)} = -\mathbf{a}_\beta(D_{pq}^2)\,\mathbf{u}_0^{(s)} - b_\beta(D_{pq}^3)\,w_0^{(s)} + (z^+ \tau_\beta^+ - z^- \tau_\beta^-)\,\delta_s^1$$

and satisfy the "coupling" equations

$$q_{13}^{(s+2)} = \{ \partial_1 m_1 + \partial_2 m_{12} \}^{(s+2)} + (z^+ \tau_1^+ - z^- \tau_1^-)\,\delta_s^1 \quad (1 \leftrightarrow 2)$$

The fundamental equations (3.6), written in terms of forces and moments, are the same as the classical equations of the Kirchhoff–Love theory. In statics

$$\{\partial_1 q_1 + \partial_2 q_{12}\}^{(s+2)} + (\tau_1^+ - \tau_1^-)\delta_s^1 = 0 \quad (1 \leftrightarrow 2)$$
$$\{\partial_1 q_{13} + \partial_2 q_{23}\}^{(s+2)} + (\sigma^+ - \sigma^-)\delta_s^0 = 0$$
$$\{\partial_1^2 m_1 + 2\partial_{12}^2 m_{12} + \partial_2^2 m_2\}^{(s+2)} + \operatorname{div}(z^+\boldsymbol{\tau}^+ - z^-\boldsymbol{\tau}^-)\delta_s^1 + (\sigma^+ - \sigma^-)\delta_s^0 = 0$$

5. All the preceding arguments were phrased for an arbitrary position of the origin along the vertical; they yield consistent equations for the linear and transverse displacements. We will now analyse the possibility of a special choice of system of coordinates and separation of the problems. To that end we will need the complete expressions for the fundamental operators

$$\begin{aligned} \mathbf{A}_1 &= -\mathbf{i}_1(D_{11}^1\partial_1^2 + 2D_{16}^1\partial_{12}^2 + D_{66}^1\partial_2^2) - \mathbf{i}_2(D_{16}^1\partial_1^2 + (D_{12}^1 + D_{66}^1)\partial_{12}^2 + D_{26}^1\partial_2^2) \\ B_1 &= D_{11}^2\partial_1^3 + 3D_{16}^2\partial_1^2\partial_2 + (D_{12}^2 + 2D_{66}^2)\partial_1\partial_2^2 + D_{26}^2\partial_2^3 \quad (1 \leftrightarrow 2) \end{aligned} \tag{5.1}$$

Generally speaking, none of the terms in (5.1) will vanish and the proportions among them may be arbitrary. That is the situation, for example, in the most-general anisotropic conditions or in the case of asymmetrically assembled orthotropic layers or crosswise stacking. We have thus obtained rather contradictory conditions for eliminating the operators B_1, B_2 and A from Eqs (3.6), and the following proposition holds.

Proposition 1. The problems of bending and linear tension-compression-shearing are not separable for an assembly of N anisotropic layers, arbitrarily stacked with respect to their thickness. Hence it is impossible to indicate an equivalent anisotropic monoplate with the average characteristics of the layers, as has been done for isotropically elastic layers [4].†

The physical reason for the interconnections among the problems is that the strains

$$\epsilon_{11}^{(s)} = \partial_1 u_{01}^{(s)} - z\partial_1^2 w_0^{(1)}, \quad \epsilon_{12}^{(s)} = \tfrac{1}{2}(\partial_1 u_{02} + \partial_2 u_{01} - 2z\partial_{12}^2 w_0)^{(s)} \quad (1 \leftrightarrow 2)$$

may turn out to be non-zero for any position of the longitudinal plane x_1x_2, so that Kirchhoff's first hypothesis is violated.

If the layers are symmetrically arranged, it is natural to put $z = 0$ in the middle plane. Then the sum $\Sigma_j(z_{j+1}^2 - z_j^2)F_j$ will vanish for any symbols F_j independent of the vertical coordinates. The membrane-flexural stiffnesses must be eliminated in Eqs (3.6) and in the expressions for the forces and moments; this automatically isolates the problem of bending for an assembly with undeformable middle plane.

Proposition 2. The behaviour of an assembly of $N = 2n+1$ layers, symmetrically placed with respect to their thickness, is similar to that of an anisotropic monoplate with an undeformable middle plate.

We emphasize that, unlike the situation in the hypothetical construction of equations, we are defining the complete stress tensor, including asymptotically small components, which may be used to analyse internal stresses, the strength of adherence of the layers, and so on .

We will now consider an intermediate situation. A natural test of when the problems are interconnected is the function

$$F(z_1) = \{\|B_1\|^2 + \|B_2\|^2\}^{1/2} =$$
$$= \{(D_{11}^2)^2 + (10D_{16}^2)^2 + 2(D_{12}^2 + 2D_{66}^2)^2 + (10D_{26}^2)^2 + (D_{22}^2)^2\}^{1/2}$$

which has the property that $F^2(z_0)$ is a quadratic polynomial. The vertical position of the origin is chosen subject to the condition

$$z_1 = z^-: \ \xi \equiv F(z^-) = \min F(z)$$

†See also SIMONOV I. V., Dynamical equations of the bending of thin elastic plates which are degenerately inhomogeneous with respect to thickness. Preprint No. 468, Inst. Problem Mekh., Akad. Nauk SSSR, 1990.

Proposition 3. If $\xi \ll 1$, the problems of bending and linear tension-compression-shearing of a layered assembly may be separated step by step in the following iterative procedure

$$\mathbf{D}_0 \mathbf{v}_1^{(s)} = \mathbf{R}^{(s)}, \quad \mathbf{D}_0 \mathbf{v}_{n+1}^{(s)} = -\mathbf{D}_1 \mathbf{y}_n^{(s)}, \quad \| \mathbf{D}_0 \|, \ \| \mathbf{D}_1 \| = O(1)$$

$$\mathbf{v} = (\mathbf{u}, \mathbf{u}, w)^{\mathrm{T}}, \quad \mathbf{y} = (w, w, \mathbf{u})^{\mathrm{T}}, \quad \mathbf{v}_0^{(s)} = \sum_{n=1}^{\infty} \xi^{n-1} \mathbf{v}_n^{(s)}$$

where $\mathbf{R}^{(s)}$ is the vector of loads in Eqs (3.6), $\mathbf{D}_0$ is the operator corresponding to the separate problems of bending and generalized plane-stress state, and $\mathbf{D}_1$ is the operator with membrane-flexural components

$$\mathbf{D}_0 = (\mathbf{A}_1, \mathbf{A}_2, B_* + \rho_* \partial_t^2)^{\mathrm{T}}, \quad \mathbf{D}_1 = \xi^{-1} (B_1, B_2, \mathbf{A}_*)^{\mathrm{T}}$$

An exact determination of the radius of convergence of the series in ξ requires a further analysis of the specific boundary conditions, but it is quite obvious that for sufficiently small values of ξ, the iteration procedure will converge in any reasonably chosen norm.

The case $\xi = 0$ exhausts all combinations of the parameters that admit of complete separation of the problems.

We will consider some of these special anisotropic structures.

Proposition 4. Consider a composite beam made up of orthotropic layers with principal axes in the x_1, x_2, x_3 directions, loaded along the x_2 axis. Then the problem may be separated if one chooses

$$z_1 = -\frac{1}{2D_{11}^1} \sum_{j=1}^{N} h_j H_j^* \gamma_{11}^j, \quad H_j^* = h_j + 2 \sum_{n=1}^{j-1} h_n \tag{5.2}$$

The mean stiffnesses, Young's modulus and Poisson's ratios for bending and tension-compression-shear for this choice of parameter are

$$\nu_*^0 = D_{12}^3 / D_{11}^3, \quad e_*^0 = {}^3\!/\!_2 D_{11}^3 [1 - (\nu_*^0)^2] \tag{5.3}$$

$$\nu_*^1 = D_{12}^1 / D_{11}^1, \quad e_*^1 = \tfrac{1}{2} D_{11}^1 [1 - (\nu_*^1)^2]$$

$$\gamma_{11}^j = \left\{ \frac{e_1}{1 - \nu_{12}\nu_{21}} \right\}_j, \quad \gamma_{12}^j = \nu_{21}^j \gamma_{11}^j$$

so that one obtains an analogous isotropic monobeam. There is a slight difference, in that the secondary linear forces and/or bending moments in the reduced middle plane need not vanish

$$q_2^{(s+2)} = \nu_*^1 D_{11}^1 \partial_1 u_{01}^{(s)} - D_{12}^2 \partial_1^2 w_0^{(s)}, \quad m_2^{(s+2)} = D_{12}^2 \partial_1 u_{01}^{(s)} - \nu_*^0 D_{11}^3 \partial_1^2 w_0^{(s)}$$

and in this sense the middle plane is not neutral.

Proposition 5. If the layers are transversally isotropic (z being the common axis of anisotropy), then a separate formulation of the problem is also obtained if one chooses the reduced middle plane in accordance with formula (5.2).

Indeed, assuming that e and ν are the normalized Young's moduli and Poisson's ratios in the plane of isotropy, we see that in each layer

$$\mathbf{a}_\beta = -\frac{e}{2(1-\nu)} \partial_\beta \operatorname{div} - \frac{e}{2(1+\nu)} \mathbf{i}_\beta \Delta, \quad b_\beta = \gamma \partial_\beta \Delta$$

$$\mathbf{a}_* = -\gamma \Delta \operatorname{div}, \quad b_* = \gamma \Delta^2 \quad (\Delta \equiv \partial_1^2 + \partial_2^2)$$

$$\gamma \equiv \gamma_{11} = \gamma_{22} = \gamma_{12} + 2\gamma_{66} = \frac{e}{1-\nu^2}, \quad \gamma_{12} = \nu\gamma_{11}$$

and separated equations may then be derived from (3.6). The expressions for the moments and forces in the plate cross section are also considerably simpler, and the values of the mean Young's moduli and Poisson's ratios are determined from (5.3).

These results agree with those obtained for isotropically layered plates (see the paper cited in the previous footnote), so that all cases considered in this paper in which the layer parameters are asymptotically degenerate may be generalized naturally to structures with transversal isotropy.

6. To solve static problems for an asymmetric assembly, one can introduce a displacement potential and thus make use of the theory of functions of a complex variable, which has been developed for the case of an anisotropic monoplate [7]. Suppose we have a particular solution of the problem, corresponding to specific loads in Eqs (3.6). We seek coupled non-degenerate homogeneous solutions in the following form (omitting the index s)

$$\mathbf{u} = \mathrm{Re}[\mathbf{u}^* \varphi'(\zeta_1 x_1 + \zeta_2 x_2)], \quad w = \mathrm{Re}[u_3^* \varphi(\zeta_1 x_1 + \zeta_2 x_2)] \tag{6.1}$$

where u_α^*, ζ_β are complex constants and the function φ is differentiated globally with respect to its argument (we may assume without loss of generality that $\zeta_1 = 1$, $\zeta_2 = \zeta$, $u_3^* = 1$). Then Eqs (3.6) are identically satisfied if ζ is a root of the (eighth-order) characteristic equation, and the constants u_1^*, u_2^* may be determined from the linear system

$$p^*(\zeta_1, \zeta_2) \equiv p_{33} p_0 + p_{11} p_{23}^2 + p_{22} p_{13}^2 + 2 p_{12} p_{23} p_{31} = 0 \tag{6.2}$$
$$\mathbf{P}\mathbf{u}^* = 0, \quad \mathbf{P} = \| p_{\alpha\beta} \|, \quad p^* = \det \mathbf{P}, \quad p_0 \equiv p_{11} p_{22} - p_{12}^2$$

The polynomials $p_{\alpha\beta}$ are given by

$$p_{11} = -D_{11}^1 \zeta_1^2 - 2D_{16}^1 \zeta_1 \zeta_2 - D_{66}^1 \zeta_2^2, \quad p_{12} = -D_{16}^1 \zeta_1^2 - (D_{12}^1 + D_{66}^1)\zeta_1\zeta_2 - D_{26}^1 \zeta_2^2$$
$$p_{13} = D_{11}^2 \zeta_1^3 + 3D_{16}^2 \zeta_1^2 \zeta_2 + (D_{12}^2 + 2D_{66}^2)\zeta_1 \zeta_2^2 + D_{26}^2 \zeta_2^3, \quad p_{31} = -p_{13} \ (1 \leftrightarrow 2)$$
$$p_{33} = D_{11}^3 \zeta_1^4 + 4D_{16}^3 \zeta_1^3 \zeta_2 + 2(D_{12}^3 + 2D_{66}^3)\zeta_1^2 \zeta_2^2 + 4D_{26}^3 \zeta_1 \zeta_2^3 + D_{22}^3 \zeta_2^4$$

If an undeformable "neutral" plane ($\xi = 0$) exists and the problems of bending and the generalized plane-stress state are separable, then $p_{\beta 3} \equiv 0$. The roots of the characteristic equation (6.2) also fall into two groups, corresponding to the two problems. For the bending problem $u_1^* = u_2^* = 0$ and for the plane problem $u_3^* = 0$; the substitution $\psi = \varphi'$, followed by some minor algebra, makes it possible to use well known methods [7].

Proposition 6. The characteristic equation (6.2) and the characteristic equations for the components $(\mathbf{A}_1, \mathbf{A}_2)$ and B_* of the reduced operator $\mathbf{D}_0$ have no real roots. The complex roots form conjugate pairs.

To prove this, consider the expression for the potential energy of the plate. Using asymptotic expansions, we obtain

$$\Pi^{(s)} = \frac{1}{2} \int_\Omega \{ q_1 e_{11} + 2q_{12} e_{12} + q_2 e_{22} - m_1 \partial_1^2 w_0 - 2m_{12} \partial_{12}^2 w_0 - m_2 \partial_2^2 w_0 \}^{(s)} d\Omega \tag{6.3}$$
$$\Pi = E_0 L_0^3 \epsilon^{-3} \{ \Pi^{(0)} + \epsilon^2 \Pi^{(1)} + O(\epsilon^4) \}, \quad e_{\alpha\beta} = \tfrac{1}{2}(\partial_\alpha u_{0\beta} + \partial_\beta u_{0\alpha})$$

Substituting (6.1) into (6.3), we obtain a quadratic form

$$\Pi^*(\mathbf{u}^*) = \frac{1}{2} \int_\Omega \{ p_{33} u_3^* \bar{u}_3^* - \sum_{\alpha\beta \neq 33} p_{\alpha\beta} u_\alpha^* \bar{u}_\beta^* \} |\varphi''|^2 d\Omega, \quad \zeta \in R$$

Since the energy is positive definite, this implies that $p^*(\zeta), p_0(\zeta), p_{33}(\zeta) > 0$ for $\zeta \in R$. The second part of the proposition follows from the fact that all the coefficients of the polynomials are real. Hence the operators are elliptic and the problem can be solved with the operator $\mathbf{D}_0$ by the same methods as in classical plate theory.

Finally, expressions (6.1) become

$$(\mathbf{u}_0, w_0) = \sum_{k=1}^{4} \mathrm{Re}(\mathbf{u}^* \varphi', u_3^* \varphi)_k, \quad \zeta_{k+4} = \bar{\zeta}_k, \quad \zeta_\zeta \neq \zeta_j \tag{6.4}$$

and are sufficiently arbitrary to satisfy the combined boundary conditions on the set $\partial\Omega$.

Physical considerations dictate that there should be four such conditions, most naturally formulated by combining the boundary conditions for the problems of bending and the generalized plane-stress state of the plate [5, 7–9]. The error due to the integral conditions at the ends may depend on the choice of these conditions [3], but in isotropic plates, as a rule, it does not exceed $O(\epsilon)$ outside a boundary layer. The detailed construction of the boundary layer for a layered plate and its interaction with the internal stresses requires special treatment.

In conclusion, we present an example in which the representation (6.4) is particularly simple. Consider an elliptic layered plate of asymmetric structure with a rigidly clamped contour

$$\partial\Omega:\ f\equiv(x_1/c_1)^2+(x_2/c_2)^2-1=0,\quad \mathbf{u}_0=0,\quad w_0=\partial_n w_0=0$$

The solution for a constant normal load and linear shear loads

$$\sigma^{\pm}=\text{const},\quad \tau^{\pm}=\tau_1^{\pm}x_1+\tau_2^{\pm}x_2,\quad \tau_\beta^{\pm}=\text{const}$$

is

$$u_{0\beta}=u_{\beta1}\partial_1 f^2+u_{\beta2}\partial_2 f^2,\quad w_0=u_{33}f^2,\quad u_{\alpha\beta}\in R$$

After substitution into Eqs (3.6), we obtain a linear system of fifth-order equations for the constants $u_{\alpha\beta}$. If the assembly is orthotropic and the semi-axes of the ellipse lie along the principal axes, then $u_{12}=u_{21}=0$ and the number of equations is reduced to three.

7. In the steady state, the dispersion relation for a monochromatic wave $(\mathbf{u}_0,\ w_0)\equiv(\mathbf{u}^*,\ u_3^*)\exp(i\omega t-i\eta(x_1\cos\theta+x_2\sin\theta))$ is determined by the polynomials (6.2) and becomes

$$\omega^2=\frac{\eta^4 p^*(\cos\theta,\sin\theta)}{\rho_* p_0(\cos\theta,\sin\theta)}$$

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